The Incubation Effect: How Mathematicians Recover From Proving Impasses

Milos Savic

University of Oklahoma

Savic@ou.edu

Abstract

The literature on mathematicians' actions during proving has, thus far, been primarily anecdotal. This paper reports the **observed** actions of nine mathematicians, six of whom came to an impasse while constructing proofs alone on an unfamiliar topic, from a set of notes, and with unlimited time. The existence of impasses and the actions participants took to recover from them were either directly observed from the real-time data collected (using an innovative technique) or obtained from exit interviews or focus groups. Certain times could be considered a period of *incubation*, which psychologists have defined as a "temporary shift away from an unsolved problem that allows a solution to emerge seemingly as if from no additional effort" (Sio & Ormerod, 2009, p. 94). These actions to overcome impasses, while naturally part of mathematicians' proving processes, could be discussed with students in a classroom setting to help alleviate difficulties in proving.

Keywords: proving process, mathematicians, psychology of mathematics, proof-based courses

The proving process has been examined at the university level from various viewpoints including: students' difficulties with the overall process (Moore, 1994; Weber & Alcock, 2004), students' difficulties with validations of proofs (Selden & Selden, 2003), students' difficulties with comprehension of proofs (Conradie & Frith, 2000; Mejia-Ramos, Weber, Fuller, Samkoff, Search, & Rhoads, 2010), as well as by the categorization of students' proof schemes (Harel & Sowder, 1998), that is, by the ways university mathematics students decide what is convincing and persuading. This study takes up another important aspect of the proving process that has as yet been discussed mainly anecdotally (Hadamard, 1945; Poincaré, 1946; Liljedahl, 2004), namely, how mathematicians respond to, and often overcome "getting stuck."

1. Background Literature

1.1 Impasses, incubation, and insight in psychology and neuroscience literature

In examining mathematicians' proving practices, the study reported here focused on impasses and how the mathematicians overcame those impasses, including incubation and the resulting insights. The motivation for this examination is highlighted by Sio and Ormerod (2009), who stated that, "understanding the role of incubation periods may also allow us to make use of them effectively to promote creativity in areas such as individual problem solving, classroom learning, and work environments" (p. 94). Impasses, incubation, and insight have been examined in the psychology and mathematics education literatures, mainly in analyzing problem solving, but there has been little research on them during proving. A brief discussion of these literatures provides background for the use of the terms impasse and incubation in examining and analyzing proof construction.

In the psychology literature, an impasse is defined as a state of mind where problem solving attempts cease and the impression arises that the problem is unsolvable (Glatzeder, Goel, & von Müller, 2010, p. 17). Problem solvers in the psychology literature sometimes recovered from impasses through *incubation*. Incubation, according to Wallas (1926), is the process by which the mind goes about solving a problem, subconsciously and automatically. It is the second of Wallas' four stages of creativity, which are:

- preparation (thoroughly understanding the problem),
- incubation (when the mind goes about solving a problem subconsciously and automatically),
- illumination (internally generating an idea after the incubation process, sometimes known as the Aha! experience), and
- verification (determining whether that idea is correct).

Incubation has also been described as "a gradual and continuous unconscious process . . . during a break in the attentive activity toward a problem" (Segal, 2004, p. 141). Neuroscientists have researched incubation using fMRI technology (Binder, Frost, Hammeke, Bellgowan, Rao, & Cox, 1999; De Luca, Beckmann, De Stefano, Matthews, & Smith, 2006). They have found that during incubation "the brain contains highly organized, spontaneous patterns of functional activity at rest" (Buckner & Vincent, 2007, p. 2). In addition, Smith and Blankenship (1991) stated that "the time in which the unsolved problem has been put aside refers to the *incubation time*; if insight [illumination] occurs during this time, the result is referred to as an *incubation effect*" (p. 61). It has been conjectured that this effect happens best when one takes a break from creative work (Krashen, 2001).

There seem to be at least two incubation techniques that have yielded positive effects during problem solving: deliberately taking breaks and using "low-demand" tasks. Deliberate incubation has been shown to result in a greater incubation effect than merely being interrupted during the problem-solving process: "Individuals who took breaks at their own discretion (a) solved more problems and (b) reached fewer impasses than interrupted individuals" (Beeftink, van Eerde, & Rutte, 2008, p. 362). Scientists have studied which tasks can be done during those deliberate breaks. In their meta-analysis of 29 articles covering 117 separate psychology experiments dealing with incubation, Sio and Ormerod (2009) stated that "low-demand tasks[1]" done during an incubation period yielded positive incubation effects compared to "high-demand tasks[2]": "There remains a possibility, of course, that a sufficiently light load might allow additional covert problem solving compared with a heavier task load" (p. 107).

1.2 Impasses and incubation in mathematics and mathematics education

There is an extensive problem-solving literature (e.g., Schoenfeld, 1992, Carlson & Bloom, 2005) but for this study, the concentration is on how provers handle impasses, which includes experiencing periods of incubation. To date, research on incubation during problem solving in the mathematics education literature has been sparse and primarily anecdotal. Perhaps this is because creativity, which includes incubation, has rarely been captured in mathematics education research: "[S]tudying a mathematician's or student's creativity is a very difficult enterprise because most traditional operationalized instruments fail to capture extra cognitive traits, such as

[1] Examples of low-demand tasks demonstrated in incubation experiments (Sio & Ormerod, 2009) include listening to music or memorization of a passage.

[2] An example of a high-demand task demonstrated in incubation experiments (Sio & Ormerod, 2009) included the "farm problem," which asks to "divide an L-shaped farm into four parts that have the same shape and size."

beliefs, aesthetics, intuitions, intellectual values, self-imposed subjective norms, spontaneity, perseverance standards, and chance" (Freiman & Sriraman, 2007, p. 23). Some instruments that have been used to capture creativity in mathematics education include video interviews, written work, or problem/proving sessions in front of a camera (Garii, 2002).

Mathematicians have sometimes acknowledged that preparation is a requirement for creativity. Poincaré (1958) described explicitly a time during which he experienced an insight after an incubation period. He believed that the "preparation stage" along with incorrect attempts on proofs is more useful than one usually thinks, believing it sets the unconscious mind at work. Observation of the unconscious mind is beyond the scope of this study, yet Poincaré and Hadamard (1945) both devoted many thoughts and conjectures to this.

The mathematician Mordell (1959) suggested that mathematicians need to be motivated by, and immersed in, a problem for creativity, including incubation, to occur. Byers (2007), another mathematician, described stages similar to those of Wallas (1926) and Poincaré (1958), but seemed to focus as much on finding as on proving theorems. He stated that:

> The mathematician's work can be broken down into various stages. The first involves spade work: collecting data and observations, performing calculations, or otherwise familiarizing oneself with a certain body of mathematical phenomena. Then there are the first inklings that there exists in this situation a pattern or regularity—something that is going on. This is followed by the hard work of bringing the embryo into fruition. Then, finally, when the idea has appeared, there is the stage of verification or proof.
> (pp. 196-197)

In his investigation of mathematicians' practices, Hadamard (1945) mailed surveys to mathematicians around the world to collect information on what mathematicians think they do. Nicolle, one of the mathematicians who responded to Hadamard's survey, concluded, "contrary to progressive acquirements, such an act [discovery of a solution after an impasse] owes nothing to logic or to reason. The act of discovery is an accident" (p. 19). While discovery may be an "accident," the actions taken to allow the mind to have such an accident might be quite deliberate.

1.3 The difficulty of investigating incubation

While many results and theories about incubation may be transferred from the psychology literature, the problems designed by psychologists to investigate incubation in problem solving lack mathematical rigor. The kinds of problems used in psychology experiments are often remote association tasks, in which three words are given (*electric*, *high*, and *wheel*) and the participant is to come up with a fourth word (*chair*) that can form an association among those three words (for example, by modifying each word)[3]. All of the 117 experiments considered by Sio and Ormerod (2009) in their meta-analysis of incubation studies used incubation periods of just 1-60 minutes. However, mathematicians routinely take much more time (hours, days, weeks, possibly years) to overcome impasses in their work, and their proofs tend to be considerably longer and more complex than the remote association, or other, tasks used by psychologists.

How incubation can help mathematicians is also still somewhat of a mystery. In his dissertation research on "AHA!" experiences or insight after incubation, Liljedahl (2004) used

[3] For a thorough summary of the many kinds of problems used in incubation and insight experiments in psychology, see Chu and MacGregor (2011, pp. 125-129).

interviews with mathematicians in an attempt to obtain data on insight. He tried creating an environment for mathematicians to exhibit insight, but conceded, “the clinical interview is not at all conducive to the fostering of such phenomena” (p. 49). Sriraman (2004) conjectured that “the mind throws out fragments (ideas) which are products of past experience” (p. 30). Those fragments are brought to light as insights. Hadamard (1945), Liljedahl (2004), and Sriraman (2004) uncovered, through interviews and surveys with mathematicians, some evidence that mathematicians use incubation and then experience insights when solving problems or proving theorems.

Through this study, the hope is to add to the previous literature, partly by narrowing the focus to theorem proving, while providing as realistic a proving setting as possible. Although the methodology does not account for some of Freiman and Sriraman’s (2007) “extra-cognitive” traits, the current research focused on describing reportable or observable events, as opposed to speculating on complex mental activity outside of consciousness. According to the previous research described above, the creative process often includes impasses and incubation, but there have been difficulties capturing the proving process in a naturalistic setting in real time (Liljedahl, 2004; Freiman & Sriraman, 2007). The methodology used in this study, with the appropriate equipment, can introduce researchers to different techniques for capturing mathematical actions during problem solving or proving.

2. Research Questions

Research on mathematicians’ practices has been conducted in order to guide teaching, since “the knowledge, practices, and habits of mind of research mathematicians are . . . relevant to school mathematics education” (Bass, 2005, p. 417). Weber (2008) stated that, “investigations into the practices of professional mathematicians should have a strong influence on what is taught in mathematics classrooms" (p. 451). Past research into mathematicians' practices include Burton (1999), Misfeldt (2003), Carlson and Bloom (2005), Wilkerson-Jerde and Wilensky (2011), and more recently, Shepherd and van de Sande (2014). This research adds to both the creativity literature and the mathematicians' practices literature.

In this study, the following specific research questions were investigated:

1. How can one create an experiment, akin to those of the psychology literature, that investigates incubation during proving?
2. What do mathematicians naturally do when they reach a proving impasse?
3. Are these actions deliberate or accidental?

While psychologists' treatments of impasses, incubation, and insight may be useful in investigating a number of instances of creativity and problem solving, especially simpler instances, analyzing the construction of original proofs in mathematics seems to call for some modification of the definitions of impasse and incubation.

3. Impasses and Incubation in Problem Solving

For this study, the definition of a *proving impasse* is a period of time during the proving process when a prover feels or recognizes that his or her argument has not been progressing fruitfully and that he or she has no new ideas. What matters is not the exact length of time, or the discovery of an error, but the prover's awareness that the argument has not been progressing and requires a new direction or new ideas that are not forthcoming. Mathematicians themselves often colloquially refer to impasses as "being stuck" or "spinning one's wheels." This is different from simply "changing directions," when a prover decides, *without much hesitation*, to use a different method, strategy, or key idea, and the work on the argument on the same proof continues.

The definition of *incubation* is a period of time, following an attempt to construct at least part of a proof, during which similar activity (e.g., work on that same proof) does not occur. Similar activity may include different approaches to a proof without break, but does not include working on other problems or proofs or stepping away from a proof altogether. After incubation, a prover might have an insight, that is, the generation of a new idea. That insight might be helpful and might move the argument forward, or might not help further the argument. For some of the participants' major incubations described herein, resulting insights occurred, they were helpful, and they moved the argument forward. However, in future studies, in which the participants might be less skilled or the proofs might be more difficult, incubation might be less fruitful. Also, all but one of the incubations described herein were purposeful, but with future studies in mind, purposefulness is not included in the meaning of incubation. A lengthy proving process might entail several impasses and a number of incubation periods with subsequent insights, only some of which ultimately contribute to the final proof.

4. Methodology

4.1 Participants and materials

Nine Ph.D. mathematicians (three algebraists, three topologists, two analysts, and one logician) agreed to participate in this study on proving. The eight male mathematicians and one female mathematician were given pseudonyms, Drs. A-I, in order of participation. All were granted Ph.D.'s from North American universities. They were selected by the researcher to participate based on convenience and rapport. All were from one southwestern U.S. Ph.D.-

granting university and eight of the nine mathematicians were then currently active in their research.

They were provided with notes on semigroups (Appendix A) containing ten definitions, seven requests for examples, four questions to answer, and 13 theorems to prove. The notes were a slight modification of the semigroups portion of the notes for a "proofs course"[4] for beginning mathematics graduate students who lacked confidence in their proving skills. All of the participants in the study were unaware of the notes' origin. The slight modification was merely a renumbering for easy access to what was required of participants. Every item that required an answer or proof was renumbered (1-22), and every definition was lettered (A-J).

The topic, semigroups, was selected because the mathematicians would hopefully find the material easily accessible. Although semigroups may sometimes be considered a main topic in abstract algebra, in the researcher's experience with courses at multiple U.S.-based universities, groups, rings, and fields are the main foci of most Abstract Algebra courses, typically without mentioning semigroups. There are two theorems towards the end of the notes (Theorems 20 and 21, Appendix A) that have in the past caused substantial difficulties for beginning mathematics graduate students. Furthermore, during their exit interviews, two mathematicians proffered that the choice of semigroups had been judicious, because they had been able to grasp the definitions and concepts quickly, and because at least one of the theorems (20 or 21) had been somewhat challenging to prove. For reference, indications of how one might prove both theorems are given in Appendix B.

[4] For a description of the "proofs course," please refer to Selden, McKee, and Selden (2010, p. 207).

4.2 Data collection

The data collection began with four mathematicians separately writing proofs on a tablet PC. Later five additional mathematicians wrote their proofs with a Livescribe™ pen and special recording paper. The participants were contacted in advance to arrange a convenient meeting time. The researcher provided the technology, gave a 10-15 minute tutorial on how to operate it, and asked each participant to answer or prove each numbered theorem, question, or request for examples. The mathematicians were requested to do what they would do naturally with the numbered items. They were not explicitly advised to speak to the technology, but were informed that any sounds would be recorded along with their writing.

With the tablet PC group, the researcher approached each mathematician separately to explain how to use the hardware and the software. The researcher explained how to use the stylus that came with the tablet PC and how to turn the tablet PC around in order to be able to write on the screen. There were two software programs on the tablet PC that the mathematicians worked with: CamStudio™ screen-capturing software, which captured time-and-date stamps for each session, and Microsoft OneNote™, which was the space on which the mathematicians wrote their proof attempts. The mathematicians (Drs. A-D) each kept the tablet PC for a period of 2-7 days. After the tablet PC was returned, the researcher analyzed the screen captures (resembling small movies in real time) and the mathematicians' proof writing attempts. All proof writing attempts on OneNote were exported as PDFs for analysis. One or two days after this initial analysis of each mathematician's work, the exit interview was conducted, during which the researcher asked about their proofs and proof-writing. The exit interview questions can be found in Appendix C.

The Livescribe™ pen group consisted of five additional mathematicians (Drs. E-I). The researcher spoke with each of these participants separately to explain how to use the Livescribe™ pen and special paper. The Livescribe™ pen captures both audio and real-time writing using a camera near the end of the ballpoint pen. When one presses the pen on the "record" square at the bottom of the special paper, it goes into audio record mode, which then allows for the real-time capture of any writing or speaking. The pen can be stopped by a "stop" button, and all proving periods are time-and-date stamped. Uploading the pen data to a computer goes through the Livescribe™ software, and the researcher exported each participant's collected proving periods together in one PDF file called a "pencast." The researcher was then able to replay the pencast to see the exact sequence of events. The mathematicians each kept the Livescribe™ pen and paper for a period of 1-10 days. The work of each mathematician was collected and analyzed for a period of 1-2 days, and then the researcher conducted an exit interview with each of the five mathematicians. The questions for this group of participants were the same as those for the tablet PC group (see Appendix C).

Initially, the tablet PC was used for data collection with the idea of allowing mathematicians time and space to prove at their leisure. The switch from tablet PC to Livescribe™ pen was implemented for several reasons. First, tablet PCs are relatively more expensive than Livescribe™ pens and the corresponding paper. Second, the size of a "movie file" for a tablet PC screen capture of 16 minutes is one gigabyte, whereas an almost five-hour session using a Livescribe™ pen is just 60 megabytes. Third, the mathematicians were much more comfortable with pen and paper than with the tablet PC and stylus. Fourth, there were no visual or auditory quality differences between the data collected using the two techniques. This allowed for a smooth transition of data collection techniques to one that the researcher felt was the most comfortable for the participants, and provided all the real-time data collection that was needed**.**

The choice of using both the tablet PC and Livescribe™ pen for capturing the mathematicians' proving processes was in response to a challenge posed by Liljedahl (2004): "a tracking method that allows for the accurate capture of the problem solving process while at the same time not restricting the participant's sincere engagement in the problem is needed [to capture insight]" (p. 203). This technique is a contribution to the field of mathematics education research for its provision of an authentic, naturalistic setting, especially with the Livescribe™ pen. The pen allowed participants to write what they pleased, when they pleased, with little training on working the equipment. As Liljedahl (2004) stated, "true problem solving requires that the participants be given lots of *time* and *space* to engage, rest, and reengage with the problem" (p. 203).

4.3 Data analysis

A participant's full proving, from the start of working on the notes until the last written moment, is defined as the participant's *proving session*. This is different than a *proving period*, which is a single recording from pressing "record" until "stop" on the tablet PC or the Livescribe™ paper. All time-and-date stamps of the proving periods were listed from the proving sessions in an Excel™ spreadsheet and the average technology times and numbers of pages used were calculated. Pages used with the tablet PC group were calculated by exporting the work from OneNote™ to a PDF and counting the pages, while the pages from Livescribe™ pen group were the actual pages of paper used.

When looking at the data, the researcher concluded that six mathematicians experienced impasses on one of Theorem 20 or Theorem 21. Instead of fully transcribing the proving sessions, timelines were constructed of four mathematicians' work on certain theorems on which they encountered impasses. A sample of a timeline of Dr. G proving Theorem 20 (of Appendix A) using a Livescribe™ pen is given in Table 1. Two of the six mathematicians had technological failures of some kind, so timelines could not be constructed for their work, but their written data was analyzed. This allowed the researcher to posit if and when these two mathematicians had come to an impasse. Consequently, the researcher asked about each posited impasse during the exit interview to obtain additional information about how that participant had handled the impasse.

Table 1: Sample timeline data for Dr. G

7:02 AM	12/3/11	2 min.	Dr. G wrote the theorem and paused for a minute and a half. Then he wrote, "Hmm . . . I'm taking a **break**, breakfast, etc. Back to this later. Must think on this."
BREAK 7:04 AM - 8:07 AM			
8:07 AM	12/3/11	3 min.	Dr. G wrote, "Ok, I thought about this while on a cold walk in the fog." He then proceeded to create an ideal $gS = \{gs \mid s \in S\}$. He correctly concluded that $gS = S$, and then claimed that there are inverses. After 30 seconds, he struck through his proof, and claimed that he "need[s] an identity, not given." (This is an **impasse** that Dr. G experiences, with a subsequent **incubation** period.)
BREAK 8:10 AM - 9:44 AM			
9:44 AM	12/3/11	15 min.	He was "suspicious" whether the theorem is true and

			struggled for a counterexample. He talked about what the counterexample would have to satisfy, and then moved on to Theorem 21. He then stated an incorrect counter-example for Theorem 21, and moved on to Question 22. He answered all of Question 22 correctly, and then stated, “I should be returning to Theorem 20 which is the remaining outstanding thing. But I think I need a **break** to think about it.” (This is an external acknowledgment of **incubation.)**
BREAK 9:59 AM - 10:08 AM			
10:08 AM	12/3/11	11 min.	He used his previous ideal (gS) and manipulated it the correct way to obtain an identity and inverses.

These timelines served as a concise summary of the data collected. Dr. G did the above underlining on the paper, and the researcher highlighted the bold words for emphasis regarding possible incubation periods. If there was a certain item that needed to be examined more thoroughly, the researcher looked at the timeline, which allowed for quick access to the precise date and time in the original proving session. Also, doing this exposed the possible impasses and incubation periods. Dr. G had impasses beginning at 8:10 AM and again at 9:59 AM. In both cases, he acknowledged the need for a break since he may have had no new ideas to write down. This resulted in breaks from 8:10 AM to 9:44 AM and again from 9:59 AM to 10:08 AM. Both breaks were asked about during Dr. G’s exit interview since they could be considered incubation periods.

Looking at such timelines suggested ways a participant recovered from an impasse. For example, Dr. G took breaks, which included walking, according to his 8:07 AM session. Also, in the 9:44 AM to 9:59 AM session, he moved forward with the notes to possibly generate new ideas. Both actions were exposed when analyzing Dr. G's live data, and the timeline provided a short summary of what happened.

Time and date stamps were written on a piece of paper prior to each exit interview and these were shown to the mathematicians. This list of times allowed the interviewed mathematician to reflect on what happened during times in between the date and time stamps. Certain breaks in the proving sessions were conjectured to be incubation periods, due to the proving attempts and the content before and after the breaks. This enabled the researcher to direct the exit interview towards those times in the proving sessions. For example, knowing when an action occurred in the proving process allowed the researcher to navigate either the video or the Livescribe™ pencast to the exact time and allowed the mathematicians to relive what had happened, which helped them express additional thoughts and ideas about what he or she had been thinking at the time. For example, in his subsequent exit interview, Dr. G stated that he had had breakfast and conversation with his wife, and also acknowledged that he had been consciously thinking about the problem during 8:10 AM to 9:44 AM.

4.4 Conduct of the focus groups

Two focus group interviews were conducted by the researcher with two assistants who ran the video camera and took notes. The first focus group interview was comprised of Drs. A, B, C, and D and was 50 minutes in length. These were the mathematicians who used the tablet PC for their proofs. The interview was semi-structured and considered all the questions in Appendix C. The second focus group consisted of Drs. F, G, H, and I, mathematicians who used the Livescribe™ pen for their proofs. Dr. E was invited, but was unable to attend due to other commitments. This focus group interview was one hour and ten minutes in length and used the same questions. Forty minutes into this interview, the researcher presented all four mathematicians with two attempted proofs of Theorem 20 (Appendix A) for their consideration. Two mathematics graduate students had constructed these proofs for another study, but the mathematicians were not told of their origin. The mathematicians' consideration of those proofs is not reported in this study.

4.5 Handling Difficulties with Technology

Four of the nine mathematicians (Drs. B, D, E, and I) who participated in the study had difficulty with the technology and thus did not produce "live" data. Difficulties included not loading certain programs correctly, not remembering to press the record button, and computer error with installing CamStudio™ software. However, all four of these provided good written data, whether it was with the tablet PC on OneNote™ or with writing on the Livescribe™ paper without audio/video recording. From this data the researcher was still able to conclude that some of these mathematicians had impasses because they were candid in writing all their work, including crossing out failed attempts. They also provided informative descriptions of their thought processes (when prompted) during the exit interviews. Figure 1 shows an example of one mathematician's candidness on the tablet PC without having live data.

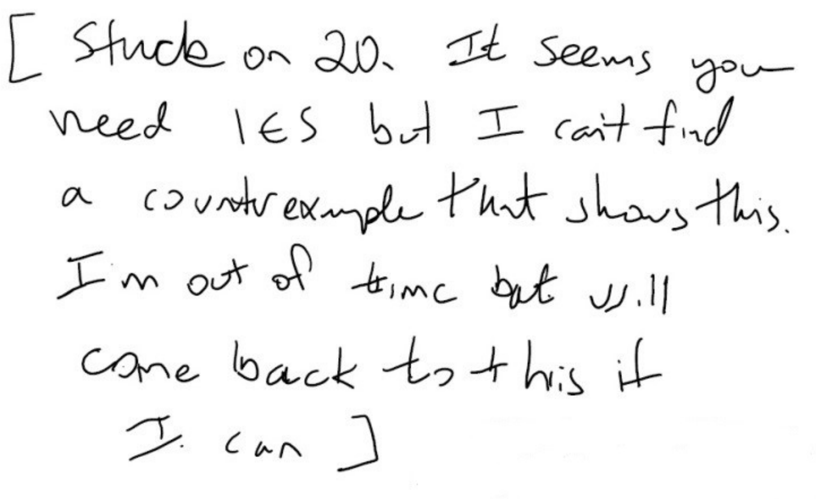

Figure 1: Dr. B's written data

5. Results

5.1 Summary data

The average total work time on the technology per mathematician was 2 hours and 5 minutes, with a range of 1 hour and 21 minutes to 2 hours and 50 minutes. This time was calculated by adding the proving periods of their actual work, obtained from the time-and-date stamps. The average time from the first "clocked in" time-and-date stamp until the last "clocked out" time-and-date stamp was 19 hours, 56 minutes, with a range of 1 hour 21 minutes to 2 days, 5 hours and 24 minutes. The average number of pages written was slightly under 13, with a range of 8 to 22. These three statistics allow one to conclude that the mathematicians expended considerable effort on proving the theorems and producing examples. Six of the nine mathematicians had impasses when proving one of the final two theorems (Theorems 20 and 21 in Appendix A). All of the mathematicians correctly proved most of the 11 other theorems very quickly.

5.2 Observed impasses by mathematicians and subsequent actions to overcome them

Below are descriptions of impasses on either Theorem 20 or Theorem 21 exhibited by the six mathematicians and the subsequent actions they took to overcome them.

5.2.1 Dr. A

Dr. A had attempted a proof of Theorem 21, "If S is a commutative semigroup with a minimal ideal K, then K is a group," for 4 minutes and 50 seconds continuously before pausing at a statement he made. The pause lasted for 1 minute and 45 seconds, and then he moved on to Question 22 requesting examples. Though there was no verbal or written acknowledgment, Dr. A's movement to the example request was one way to acknowledge an impasse.

After he had finished the request for examples, he then scrolled back up to his first attempt to prove Theorem 21, which he then erased, and also scrolled back up to his earlier correct proof of Theorem 20. Attempting to understand his proof of Theorem 20 is the second action Dr. A took to overcome his impasse. Dr. A's proving session ended at 4:17 PM that day, and started again at 11:07AM the next day. The overnight break from his attempted proof of Theorem 21 was the third action he took to overcome his impasse.

Dr. A's first attempt on Day Two used mappings and inverse mappings of elements of the minimal ideal that he had assumed during his first proving attempt. This was interesting because inverse mappings are not discussed in the notes. In his exit interview, Dr. A stated that he had used his own research to help overcome the impasse. He had written on the tablet, "I don't know how to prove that K itself is a group. For example, I don't know how to show that there is an element of K that fixes k_0 (an element of the minimal ideal K)." This was the first written acknowledgment of the impasse. For 50 minutes thereafter, he tried the mapping argument with no success. After that, there was a 27-minute break, and Dr. A successfully proved Theorem 21 at 12:55 PM on Day Two.

In his exit interview, Dr. A also indicated how he consciously generally recovers from impasses: He prefers to get "un-stuck" by walking around, but distractions caused by his departmental duties also help. That is, he often takes a break from his creative work by purposely doing something unrelated. Dr. A exhibited four distinct observed actions to overcome his impasse: moving on in the notes, looking back at his previous proof attempts, taking a break, and applying new ideas from his own research to the proof.

5.2.2 Dr. B

Dr. B experienced an impasse on Theorem 20, "If S is a commutative semigroup with no proper ideals, then S is a group." With Dr. B, there were no screen captures, but his written proof attempts were very detailed and the exit interview was also very informative. He wrote, "Stuck on [Theorem] 20. It seems you need [to hypothesize] $1 \in S$, but I can't find a counterexample to show this [that the theorem is false]" (see Figure 1). He thus acknowledged that he was at an impasse.

Dr. B next moved on to the final theorem (Theorem 21), proved it correctly, and then crossed out his proof, possibly because he had used his as yet unproved Theorem 20. After that, he moved on to the final request for examples (Question 22), explaining in his exit interview, "I moved on because I was stuck [on Theorem 20]...maybe I was going to use one of those examples...I might get more information by going ahead." Dr. B's next approach was to attempt to create counterexamples for Theorem 20. After considering his candidates for counterexamples for some time and being interrupted by taking his family to lunch, Dr. B proved both Theorems

20 and 21 correctly. In his exit interview, he stated, "I probably spent 30 minutes to an hour trying to come up with a crazy example. I went to lunch and while I was at lunch, then it occurred to me that I was thinking about it the wrong way. So I went back then and [the proof] was quick." Dr. B's actions to overcome his impasse included moving on to the next theorem, creating counterexamples, and being interrupted by his family, where, at lunch, he had an insight that turned out to be useful for furthering his proof.

5.2.3 Dr. C

Dr. C wrote down the statement of Theorem 21 incorrectly on the tablet as "If S is a commutative semigroup with a minimal ideal K, then S is a group." He then tried to prove that S, rather than K, is a group using the proving technique of creating ideals that he had used successfully in his previous proof of Theorem 20. Indeed, if what he wrote had been correct, then he could have used Theorem 20. He then had a 6-minute gap during which there was silence and no screen movement, and then he acknowledged his impasse, writing on the screen, "I'll come back to this one." He next tried to get out of the impasse by moving on to the request for examples (Question 22). Finally, Dr. C claimed in his writing that he had a counterexample, which he did for the incorrect statement that he had initially written down, but not for how Theorem 21 was stated in the notes. In his exit interview, Dr. C was re-read the theorem by the researcher and noticed his mistake. Thirty minutes after the interview, he called the researcher into his office and showed him on a chalkboard in his office the correct proof. Dr. C had two distinct actions to recover from his impasse: moving on in the notes and attempting to create counterexamples.

5.2.4 Dr. G

Dr. G claimed that for the proof of Theorem 20 he "need[ed] an identity, not given." He acknowledged his impasse then, and immediately took a break of 1 hour and 30 minutes (for reference, see Table 1). The break he took was the first action Dr. G did to get out of his impasse. Dr. G stated in his exit interview, "When I'm stuck, I often feel like taking a break. And indeed, you come back later and certainly for a mathematician you go off on a walk and you think about it." He next attempted to create counterexamples unsuccessfully. This attempted creation of counterexamples led the researcher to believe that Dr. G had not felt that the theorem was stated correctly. Dr. G stated in his exit interview that he had moved on to the next theorem and the request for examples, yet another action to overcome his impasse. After another 10-minute break, he successfully proved the theorem. For this single impasse, Dr. G tried three distinct actions to overcome the impasse: taking a break, attempting to create counterexamples, and moving on in the notes.

5.2.5 Dr. H

Dr. H first approached a proof of Theorem 20 by incorrectly assuming that there was an identity element in S. He was satisfied with his proof of Theorem 20 at the time and then used Theorem 20 to prove Theorem 21. He stated and wrote with the Livescribe™ pen, "What about the non-commutative case?" Dr. H wanted to, in his spoken words during the proving process, "mimic what we had before," so next tried to find out where the commutative hypothesis was used. While finishing up his exploration of Theorem 20 in the non-commutative case, he stated and wrote, "Back to Theorem 20 – we assumed an identity, but this was not given." Dr. H then wrote, "Revisit tomorrow." After that, he drew a question mark over the period, because he apparently wanted to reason that the identity was generated from the definition of commutative. He next re-read out loud the definition of commutative semigroup, and immediately stated, “So why must it have an identity? ... I need to prove that it has an identity.” After 1 ½ minutes of exploration, he stated, “Let’s just finish…the notes for now, and we’ll come back to it.” Dr. H was acknowledging an impasse, and his way of getting out of it was to finish up the rest of the notes. Dr. H finished the notes at 8:42 PM of that day and revisited the proof of Theorem 20 at 9:05 AM the next day, stating into the pen that he had “thought about it overnight.” He then correctly proved Theorem 20. His two observed actions were to move on in the notes and revisit the problem the next day.

Even though Dr. H had an unfinished proof of Theorem 20 before he explored the non-commutative case, he did not himself acknowledge an impasse at that point in time. Dr. H only experienced an impasse later when he acknowledged that he needed to prove an identity exists.

5.2.6 Dr. I

The last mathematician that participated in the study, Dr. I, also had an impasse on Theorem 20. Dr. I attempted to prove that S was a group by creating a principal ideal sS so that, since S has no proper ideals, $sS = S$. He did not have synchronized audio and video data, but in his writing he stated that, “Problem: show that $e_s t = t e_s = t$.” His attempt was to show that e_s is an identity element by taking another element t in S and showing that e_s leaves t unchanged. Production of an identity element is one requirement in order to show that S is a group. After a few more written lines of investigating the “problem,” he wrote, “Is this useful? Come back later.” Dr. I then moved on to Theorem 21 and the request for examples. He stated in his exit interview that if it had not been for other obligations, he would “have fiddled around with the proof for several days," acknowledging that this is how he approaches his own research. The observed action Dr. I took was moving on to a request for examples (Question 22) in the notes.

The common observed actions that the six mathematicians took included: moving on in the notes, taking breaks, creating examples and counterexamples, and looking back to previous proofs in order to apply those proving techniques. Dr. A also used some ideas from his own research to try to overcome his impasse.

5.3 Other actions by mathematicians to overcome impasses

From an analysis of the data from the tablet PCs or the Livescribe™ pens and the exit interviews, the researcher was able to ask the other participating mathematicians which actions they used to recover from impasses that had not been observed in the proving sessions or were actions that they use in their own research to overcome impasses. These were:

(a) Using a (mental) database of proving techniques: One of the mathematicians, Dr. F, stated that she has a (mental) database of proving techniques in her head.

> Your brain is randomly running through arguments you've seen in the past . . . standard techniques that keep running through my head, sort of like downloading a whole bunch at the same time and figuring out which way to go. (Dr. F)

(b) Doing other mathematics: Some mathematicians indicated that they might go to another project to help them overcome proving impasses.

> What I try to do is to keep three projects going . . . I make them in different areas and different difficulty levels . . . . (Dr. E)

(c) Doing tasks unrelated to mathematics: This is the second non-mathematical action unrelated to an impasse. This action was also perhaps the most unusual, and Dr. E seemed slightly embarrassed when he reported the action to the researcher.

> Yeah I'll do something else, and I'll just do it, and if there's a spot where I get stuck or something, I'll put it down and I'll watch TV, I'll watch the football game, or whatever it is, and then at the commercial I'll think about it and say, 'yeah that'll work'. . . . (Dr. E)

(d) Sleeping on it: The last action to overcome an impasse seemed to be the easiest for a mathematician. Proving can involve mental exhaustion, so resting can help one's exploration for new ideas.

> It often comes to me in the shower . . . you know you wake up, and your brain starts working and somehow it [an insight] just comes to me. I've definitely gotten a lot of ideas just waking up and saying 'That's how I'm going to do this problem.' (Dr. F)

6. Discussion

Six of the nine mathematicians in this study exhibited impasses and recoveries from those impasses, including some recoveries attributable to incubation. All of the mathematicians attempted both Theorems 20 and 21. It seemed that most of the mathematicians approached the notes as a challenge or a game to be completed. For example, Dr. H stated in his exit interview that, "it was kind of fun…some [of the proofs] were kind of basic, that's a standard kind of proof, and others were… 'That's kind of different.'" Dr. C exhibited another example of the need for completion during the exit interview when he realized that he had misread Theorem 21.

Thirty minutes after the exit interview, Dr. C called the researcher into his office to show a full proof of Theorem 21 on the chalkboard.

In the focus groups, the mathematicians also discussed methods of impasse recovery and what amounts to incubation (that may occur independent of an impasse). They all did this in a relaxed, assured way, not like someone discussing something unfamiliar, but rather like someone discussing practices built up over some time. Furthermore, they mentioned general benefits that appear to go beyond just restarting an argument, such as clearing the mind or developing more understanding of the theorem. Dr. A stated, "I do have a belief that if I walk away from something and come back it's more likely that I'll have an idea than if I just sit there." In a somewhat similar vein, Dr. F offered the following in her exit interview: "You just come back with a fresh mind. [Before that] you're zoomed in too much and you can't see anything around it anymore." These remarks indicate that some mathematicians take deliberate actions to overcome impasses and also to improve the breadth or quality of their perspectives. This result seems to agree with the neuroscience literature, which concluded that incubation might contribute to a person's processing the past, present, and future all at once (Buckner & Vincent, 2007).

One sees that there might still be conscious thought about the current mathematical problem during a break that may not be considered incubation. During one focus group interview, Dr. G stated:

> When we are working on something, we are usually scribbling down on paper. When you go take a break, . . . you are thinking about it in your head without any visual aids . . . [walking around] forces me to think about it from a different point of view, and try different ways of thinking about it, often global, structural points of view.

He stated that there is no “scribbling on paper” when one “take[s] a break.” Doing this, that is, thinking more generally, he believed, might assist in understanding the structure of a problem or even an area of mathematics.

Data collection using the tablet PCs or Livescribe™ pens may have some limitations. There were a number of instances in which impasses and recoveries from them might have occurred in a way that could not have been easily observed. For example, all of the mathematicians reported that when they first received the notes they immediately read them to estimate how long the proofs might take, but none started proving right away. In addition, there were substantial time periods during the proving sessions when nothing was recorded, and there were also substantial gaps between the “clock in” and “clock out” times during the proving sessions. When the mathematicians next “clocked in” after having left a proof attempt without finishing it, they almost always had a new idea to explore.

In answering the three research questions, the methodology used in this study, even with the limitations above, is one attempt at investigating incubation during proving. The results in Section 5 attempt to chronicle mathematicians’ observed actions to overcome impasses in the proving process, such as taking a walk or going to lunch. Finally, through both exit interviews and focus group interviews, the researcher found that these actions were both deliberate (e.g., taking a walk) and accidental (e.g., going to lunch) in attempting to have successful insights, although some deliberate actions seemed to have been personally developed over time by mathematicians.

7. Education Implications

The above Results and Discussion sections suggest that proving impasses, recoveries from them, incubation, insight, and the ability to deal with unfamiliarity are all a significant part of doing mathematics, and in particular, of constructing proofs. Thus, it is worth examining how these might be taught. The ways of doing this have yet to be examined in detail. However, there is support from the psychology literature about the positive effects of incubation in the classroom. Sio and Ormerod (2009) cited four articles where "educational researchers have tried to introduce incubation periods in classroom activity, and positive incubation effects in fostering students' creativity have been reported" (p. 94). For example, Medd and Houtz (2002) examined facilitating incubation in creative writing using a 10-minute incubation session in between creative writing exercises. During the 10-minute incubation session, the students in the experimental group were told to think about the assignment because they were going to add more stories to their initial attempts. Using judges on creative writing, the students in the experimental group performed better than the control group, who had the 10-minute incubation session with no prompting, or the other treatment group that did not have an incubation session.

Extraneous attributes may influence undergraduate students in proof-based courses such as abstract algebra or real analysis. They may come into the university with pre-conceived notions of how to approach mathematics. In fact, out of 206 high-school responses to the question, "How long should it take to solve a typical [mathematics] homework problem?" the average time indicated was just under 2 minutes (Schoenfeld, 1989, p. 345). Some instructors demonstrate proofs in class with little regard for how much toil past mathematicians took to create such theorems and proofs. Included in the toil were impasses and actions to overcome them. Since students may attempt to mimic their instructors (Selden, Benkhalti, & Selden, 2014), telling students what mathematicians do when they "get stuck" might help them when they have "no idea what to do next."

Factors such as grading are also influential in classrooms. Perhaps if an instructor were to value students' evolving work, including mishaps and other tentative thoughts, they might, in turn, value their unsuccessful proving attempts for what they can learn from them. Dr. G evaluated all his incorrect attempts and tried to make sense of the situations in his uncompleted proofs. Providing constructive feedback on students' proving attempts might result in having them pay the same attention to what can be learned from failed attempts. Valuing proving attempts might also allow the students to be more comfortable with attempting proofs multiple times and re-evaluating their work.

These are just suggestions for pedagogical changes for introducing the value of impasses and incubation to students. There is no claim that all a novice prover must do is to emulate an expert prover's actions to overcome impasses. However, do novice provers **know** what mathematicians endure when proving, including their impasses? When do novice provers begin developing actions to overcome their impasses? Studying which actions instructors undertake to acknowledge their students' proving impasses is one of a few research topics in need of further investigation.

8. Future Research

Using Livescribe™ pens and the corresponding special paper for gathering real-time data can provide a naturalistic setting for provers. Unlike Poincaré (1958), who reflected on and described his own actions to overcome impasses, and Hadamard (1945), who used surveys to collect data on mathematicians' recollections on their impasses and incubation, the data collection technique of tablet PCs and Livescribe™ pens used in this study captured the real-time impasses and actions to overcome them. In addition, periods of incubation were also captured during the mathematicians' exit interviews. One might ask:

- How can we use this data collection technique in the classroom?
- Would it benefit students to have Livescribe™ pens with which to do their homework so that instructors could analyze their real-time proving processes?

There may also be affective issues with recovery from impasses which were not captured by the data collection techniques used in this study. For example, Movshovitz-Hadar and Hadass (1990) who considered "cognitive conflicts" in the setting of pre-service mathematics teachers, stated, "[the participants] show expressions of curiosity arousal, and expressions of an inner drive to resolve, as well as expressions of frustration, expressions of satisfaction with coping with inability to proceed, and expressions of content with feeling self-confident about a shaky state" (p. 276). Most of these affective states were not observed in the collected data simply because the design of the study included establishing the most naturalistic environments, thereby eliminating time constraints and interview-like settings. Hence, one could ask:

- What can be done to observe affective instances during proving while still maximizing the naturalistic environment?

9. Conclusion

During the proving of many non-routine theorems, impasses are inevitable. If not, there would be no more open conjectures in mathematics. While previous studies in the mathematics education literature used self-reflections to find what mathematicians do, this study employed a data collection technique that captured real-time actions to overcome impasses during proving. The mathematicians in this study often had certain actions they used for overcoming impasses, that was, for the most part, incubation-related and personal. Also, they utilized their unsuccessful proving attempts by carefully evaluating what was causing the impasse. Acknowledging impasses and incubation in the classroom by grading (i.e., marking) and explicitly valuing proving processes might be one way to get students to begin creating personal actions for overcoming impasses. If nothing else, instructors candidly discussing the difficulties with the finished proofs presented in proof-based courses may relieve students that do not know the amount of time and thought many proofs need.

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Appendix A: Sample of the Notes for Participants

Below is an abbreviated version of the notes supplied to the participants, with many of the relevant definitions and theorems to help prove Theorems 20 and 21.

Definition A: A semigroup $(S,\cdot)$ is a nonempty set S together with a binary operation $\cdot$ on S such that the operation is associative. That is, for all a, b, and $c \in S$, $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.

Definition B: A nonempty subset A of a semigroup S is called a left ideal [right ideal, ideal] of S if $SA \subseteq A$ $[AS \subseteq A, AS \cup SA \subseteq A]$ where $SA = \{sa | s \in S \text{ and } a \in A\}$.

Definition D: A semigroup S is called commutative or Abelian if, for each a and $b \in S$, $ab = ba$.

Definition F: An element 1 of a semigroup S is called an identity element of S if, for each $s \in S$, $1s = s1 = s$. (Other symbols, such as "e", may be used instead of "1" to represent an identity element.)

Theorem 11: A semigroup can have at most one identity element and at most one zero element.

Definition J: A semigroup G is called a group if G has an identity 1 and if for each $g \in G$ there is a $g' \in G$ such that $gg' = g'g = 1$.

Theorem 20: If S is a commutative semigroup with no proper ideals, then S is a group.

Theorem 21: If S is a commutative semigroup with a minimal ideal K, then K is a group.

Question 22: For each of parts a, b, and c are the two semigroups isomorphic? Prove you are right.

(a) $(\mathbb{Z}, +)$ where $\mathbb{Z}$ is the integers and $+$ is ordinary addition. $(2\mathbb{Z}, +)$ where $2\mathbb{Z}$ is the even integers and $+$ is ordinary addition.

(b) $(\mathbb{R}, +)$ where $\mathbb{R}$ is the real numbers and $+$ is ordinary addition. $((0, \infty), \cdot)$ where $(0, \infty)$ is the positive real numbers and $\cdot$ is ordinary multiplication.

(c) $(L, \cdot)$ where $L = \{0,1,2,3,4\}$ and for $x, y \in L$, $x \cdot y = x$. $(\mathbb{Z}_5, \cdot)$ where $\mathbb{Z}_5 = \{0,1,2,3,4\}$ and $x \cdot y$ means "$xy \ mod$ 5", i.e., ordinary multiplication minus (whole) multiples of 5. For example, $4 \cdot 4 = 16 - (3 \times 5) = 1, 3 \cdot 4 = 12 - (2 \times 5) = 2$, and $3 \cdot 3 = 9 - 5 = 4$, but $2 \cdot 2 = 4$.

Appendix B: Hypothetical proving processes for Theorems 20 and 21

Theorem 20, "If S is a commutative semigroup with no proper ideals, then S is a group," is usually proved by showing that S has the two additional properties that distinguish a semigroup from a group: existence of an identity and inverses for every element of S. Since S is a semigroup, S is not an empty set, hence there exists an element $s \in S$. One common proof relies on showing that sS is an ideal, and since S has no proper ideals, $sS = S$. From this set equality, a prover can manipulate the element equations that arise. For example, $se = s$ for some $e \in S$.

Hence, e is an identity for the element s, so one must prove that e is the identity for all elements of S. Also, to prove inverses exist for all elements, one can see that given $e \in S$ is the identity element, then $st = e$ for some $t \in S$, and so $s^{-1} = t$, and the proof is concluded.

For Theorem 21, "If S is a commutative semigroup with a minimal ideal K, then K is a group," a prover must realize that there are two kinds of ideals, ideals of K and ideals of S. Once one shows that K is a commutative semigroup with no proper ideals of K, then one may then use Theorem 20 to finish the proof. One may also use the same technique of the proof of Theorem 20 (assuming $k \in K$ since K is a non-empty ideal of S) to prove Theorem 21.

Appendix C: Interview Questions

Individual Interview Questions

1. Was there anything that was particularly difficult or took you long?
2. (When there were delays) What were you thinking of at this point in time?
3. What made you think of _____ (e.g., stabilizer)?
4. What difficulties were there with the technology?
5. (With a very long delay, e.g., of several hours) What did you do in that time period? Did you think about the notes or some theorem in the notes?

Focus Group Questions

1. (Question to get them comfortable) What did you think of these notes?
2. Compare and contrast your experiences with the last 2 theorems.
3. If and when you did get stuck with these notes, how did you handle that?
4. In general, what do you do when you get stuck (in a problem, proof, with your research)?
5. Is there anything else you do or think about when attempting to prove theorems?